Determinants of box products of paths

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Abstract

Suppose that G is the graph obtained by taking the box product of a path of length n and a path of length m. Let $\mathbf{M}$ be the adjacency matrix of G. If $n = m$, H.M. Rara showed in 1996 that $\det(\mathbf{M}) = 0$. We extend this result to allow n and m to be any positive integers, and show that

$$\det(\mathbf{M}) = \begin{cases} 0 & \text{if } \gcd(n+1, m+1) \neq 1, \\ (-1)^{nm/2} & \text{if } \gcd(n+1, m+1) = 1. \end{cases}$$

1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$. We define a *graph* G to be an ordered pair of sets (V, E), where V is any set and $E \subseteq \binom{V}{2}$; we refer to V as the *vertices* and E as the *edges* of G. The *adjacency matrix* of G is denoted $\mathbf{A}(G)$ and is a matrix with rows and columns indexed by V such that

$$\mathbf{A}(G)_{i,j} = \begin{cases} 1 & \text{if } \{i, j\} \in E, \\ 0 & \text{if } \{i, j\} \notin E. \end{cases}$$

Let $\mathbf{I}_n$ be the $n \times n$ identity matrix and let $\mathbf{0}_n$ be the $n \times n$ matrix of all zeros. If G has n vertices, the *characteristic polynomial* of $\mathbf{A}(G)$ is defined to be $q_G(x) = \det(\mathbf{A}(G) - x\mathbf{I}_n)$.

Suppose G_1 and G_2 are graphs with vertex sets V_1 and V_2, and edge sets E_1 and E_2, respectively. The *box product* of G_1 and G_2, denoted $G_1 \square G_2$, is the graph with vertex set $V = V_1 \times V_2$ and such that, for $i_1, j_1 \in V_1$ and $i_2, j_2 \in V_2$, $\{(i_1, i_2), (j_1, j_2)\}$ is an edge in $G_1 \square G_2$ if and only if either $i_1 = j_1$ and $\{i_2, j_2\} \in E_2$, or $i_2 = j_2$ and $\{i_1, j_1\} \in E_2$. For an in-depth look at the box product (also referred to as the *Cartesian product*) of graphs, see [1].

Let G be a graph with vertex set $[n]$ and adjacency matrix $\mathbf{A}$, and let H be a graph with vertex set $[m]$ and adjacency matrix $\mathbf{B}$. Then, the vertices of $G \square H$ can be labeled with the elements of $[nm]$, by relabeling the vertex (i, j) as $(i-1)m + j$. Under this labeling, the adjacency matrix $\mathbf{M}$ of $G \square H$ can be written as an $n \times n$ block matrix $\mathbf{M} = [\mathbf{M}_{i,j}]$, where each $\mathbf{M}_{i,j}$ is $m \times m$. Further,

$$\mathbf{M}_{i,j} = \begin{cases} \mathbf{B} & \text{if } i = j, \\ \mathbf{I}_m & \text{if } i \neq j \text{ and } i \sim j \text{ in } G, \\ \mathbf{0}_m & \text{if } i \neq j \text{ and } i \not\sim j \text{ in } G. \end{cases}$$

The $\mathbf{M}_{i,j}$ are all elements of the commutative subring S of $\mathbb{R}^{m\times m}$ generated by $\mathbf{B}$ and $\mathbf{I}_m$. Thus, if we denote the determinant over the ring S by $\det_S$, it is not hard to see that $\det_S(\mathbf{M}) = q_G(-\mathbf{B})$, so

$$\det(\mathbf{M}) = \det\left(\det_S(\mathbf{M})\right) = \det(q_G(-\mathbf{B})).$$

We now consider the case when both G and H are *paths*.

2 Paths and Products of Paths

The *path with* n *vertices,* denoted P_n, is the graph with vertex set $V = [n]$ and edge set $E = \{(i, i+1) : i \in [n-1]\}$. Let $q_n(x)$ be the characteristic polynomial of P_n. In [2], it was shown that $\det\left(\mathbf{A}\left(P_n \square P_n\right)\right) = 0$. We extend this result, and compute the value of $\det\left(\mathbf{A}\left(P_n \square P_m\right)\right)$ for all positive integers n and m. We do this first by looking at $q_n(x)$. Note that, since $\mathbf{A}\left(P_n\right)$ is a tridiagonal matrix and has a very simple structure, many of the properties, including the roots, of $q_n(x)$ are explicitly known; for example, see [3] and [4]. We will take advantage of a few particularly nice properties of $q_n(x)$. First, we will use the following theorem from [5]. We add our own corollary below.

Theorem 2.1. *For* $n \geq 2$, $q_n(x) = -xq_{n-1}(x) - q_{n-2}(x)$.

□

Corollary 2.2. *Let* $n \geq 0$. *If* n *is even,* $q_n(x)$ *is an even polynomial. If* n *is odd,* $q_n(x)$ *is an odd polynomial.*

Proof. By inspection, corollary 2.2 is true for $n \leq 2$. Assume it is true for all $n' < n$ for some $n > 2$. This implies that $q_{n-1}(x)$ and $q_{n-2}(x)$ have opposite parities as polynomials, so $xq_{n-1}(x)$ and $q_{n-2}(x)$ have the same parity, so, using theorem 2.1, we see that $q_n(x)$ and $q_{n-2}(x)$ have the same parity. The result follows. □

We will also use the following lemma; for a proof, see [6].

Lemma 2.3. *For any* $k \geq 1$, *if* $i \in [k-1]$, *then*

$$q_k(x) = q_i(x)q_{k-i}(x) - q_{i-1}(x)q_{k-i-1}(x).$$

Further, if $q_k(l) = 0$, *then the following statements are true as well.*

(a) *If* $0 \leq s \leq k$, *then* $q_{k+s}(l) = -q_{k-s}(l)$.

(b) *If* $t \geq 1$, *then* $q_{t(k+1)-1}(l) = 0$.

□

We now are ready to prove the below theorem.

Theorem 2.4. *Suppose that* $q_k(l) = 0$. *Then, for all* $a \geq 1$ *and* $0 \leq b \leq k$, $q_{a(k+1)+b}(l) = \left(q_{k+1}(l)\right)^a q_b(l)$.

Proof. Note that theorem 2.4 trivially holds when $a = 1$ and $b = 0$. Suppose $1 \le b \le k$. Applying lemma 2.3 shows that

$$q_{k+1+b} = q_{k+1}(l)q_b(l) - q_k(l)q_{b-1}(l) = q_{k+1}(l)q_b(l),$$

and thus theorem 2.4 holds when $a = 1$ and $0 \le b \le k$. Suppose it holds when $1 \le a < a'$ and $0 \le b \le k$, for some $a' > 1$. Suppose that $0 \le b \le k$. Then, by lemma 2.3,

$$\begin{aligned}
q_{a'(k+1)+b}(l) &= q_{(a'-1)(k+1)+b+k+1}(l) \\
&= q_{(a'-1)(k+1)+b}(l)q_{k+1}(l) + q_{(a'-1)(k+1)+b-1}(l)q_k(l) \\
&= q_{(a'-1)(k+1)+b}(l)q_{k+1}(l) = \left(q_{k+1}(l)\right)^{a'-1} q_b(l)q_{k+1}(l) \\
&= \left(q_{k+1}(l)\right)^{a'} q_b(l).
\end{aligned}$$

□

Label the roots of $q_n(x)$ as $l_{n,1}, l_{n,2}, \cdots, l_{n,n}$. Using our result from the previous section, $\det(\mathbf{A}(P_n \square P_m)) = \det(q_n(-\mathbf{A}_m))$. Corollary 2.2 implies that $q_n(-\mathbf{A}_m) = (-1)^n q_n(\mathbf{A}_m)$. Further, we can factor $q_n(x)$ as

$$q_n(x) = \prod_{i=1}^{n} (l_{n,i} - x) = (-1)^n \prod_{i=1}^{n} (x - l_{n,i}).$$

Thus,

$$\begin{aligned}
\det(\mathbf{A}(P_n \square P_m)) &= \det(q_n(-\mathbf{A}_m)) = \det\left((-1)^n q_n(\mathbf{A}_m)\right) \\
&= \det\left((-1)^n(-1)^n \prod_{i=1}^{n} (\mathbf{A}_m - l_{n,i}\mathbf{I}_m)\right) \\
&= \det\left(\prod_{i=1}^{n} (\mathbf{A}_m - l_{n,i}\mathbf{I}_m)\right) \\
&= \prod_{i=1}^{n} \det(\mathbf{A}_m - l_{n,i}\mathbf{I}_m) = \prod_{i=1}^{n} q_m(l_{n,i}).
\end{aligned}$$

Since, by definition, it is immediately evident that $P_n \square P_m$ and $P_m \square P_n$ are isomorphic as graphs, it follows that $\det(\mathbf{A}(P_n \square P_m)) = \det(\mathbf{A}(P_m \square P_n))$. Thus,

$$\prod_{i=1}^{n} q_m(l_{n,i}) = \prod_{i=1}^{m} q_n(l_{m,i}).$$

This leads to the following results.

Theorem 2.5. *Suppose that $n \ge 1$. Then,* $\displaystyle\prod_{i=1}^{n} q_{n+1}(l_{n,i}) = (-1)^{n(n+1)/2}$.

Proof. By inspection, theorem 2.5 is true for $n = 1$. Suppose it is true for some $n \geq 1$. Then, by lemma 2.3, $q_{n+2}(l_{n+1,i}) = -q_n(l_{n+1,i})$ for any $i \in [n+1]$, so

$$\begin{aligned}\prod_{i=1}^{n+1} q_{n+2}(l_{n+1,i}) = \prod_{i=1}^{n+1} -q_n(l_{n+1,i}) &= (-1)^{n+1} \prod_{i=1}^{n+1} q_n(l_{n+1,i}) \\ &= (-1)^{n+1} \prod_{i=1}^{n} q_{n+1}(l_{n,i}) = (-1)^{n+1}(-1)^{n(n+1)/2} \\ &= (-1)^{n+1+n(n+1)/2} = (-1)^{(n+1)(1+n/2)} = (-1)^{(n+1)(n+2)/2}.\end{aligned}$$

□

Theorem 2.6. *Suppose* $n, m \geq 1$. *Then,*

$$\prod_{i=1}^{n} q_m(l_{n,i}) = \begin{cases} 0 & \text{if } \gcd(n+1, m+1) \neq 1, \\ (-1)^{nm/2} & \text{if } \gcd(n+1, m+1) = 1. \end{cases}$$

Proof. Note that the above product is the determinant of $\mathbf{A}(P_n \underline{\mathrm{P}}_m)$, which, as discussed above, is equal to the determinant of $\mathbf{A}(P_m \underline{\mathrm{P}}_n)$. Thus, without loss of generality, we may assume that $n \leq m$. We will induct on the remainder when $m+1$ is divided by $n+1$. Suppose this remainder is 0. Then, $\gcd(n+1, m+1) \neq 1$, and $m+1 = k(n+1)$ for some $k \geq 1$, so $m = k(n+1) - 1$. Thus, by lemma 2.3, $q_m(l_{n,i}) = 0$ for $i \in [n]$, and it follows that the product of these terms is zero. This verifies theorem 2.6 for this case.

Suppose the remainder when $m+1$ is divided by $n+1$ is 1; we then have $m+1 = k(n+1)+1$ for some $k \geq 1$. Note that this implies that $\gcd(n+1, m+1) = 1$ and $m = k(n+1)$, so, by theorem 2.4, for $i \in [n]$,

$$q_m(l_{n,i}) = q_{k(n+1)}(l_{n,i}) = (q_{n+1}(l_{n,i}))^k q_0(l_{n,i}) = (q_{n+1}(l_{n,i}))^k.$$

Thus,

$$\prod_{i=1}^{n} q_m(l_{n,i}) = \prod_{i=1}^{n} (q_{n+1}(l_{n,i}))^k = \left(\prod_{i=1}^{n} q_{n+1}(l_{n,i})\right)^k = \left((-1)^{n(n+1)/2}\right)^k,$$

by theorem 2.5. Further,

$$\left((-1)^{n(n+1)/2}\right)^k = (-1)^{nk(n+1)/2} = (-1)^{nm/2}.$$

Thus, theorem 2.6 is true in this case.

Finally, suppose theorem 2.6 is true whenever the remainder when $(m+1)$ is divided by $(n+1)$ is less than r, for some $r > 1$. Then, consider any $(m+1)$ and $(n+1)$ with $(m+1)$ having remainder r when divided by $(n+1)$. It follows that there exists $k \geq 1$ such that $m+1 = k(n+1) + r$, implying that

$m = k(n+1) + r - 1$. Then, once again applying theorem 2.4,

$$\begin{aligned}\prod_{i=1}^{n} q_m(l_{n,i}) &= \prod_{i=1}^{n} (q_{n+1}(l_{n,i}))^k q_{r-1}(l_{n,i}) = \prod_{i=1}^{n} (q_{n+1}(l_{n,i}))^k \prod_{i=1}^{n} q_{r-1}(l_{n,i}) \\ &= \left(\prod_{i=1}^{n} q_{n+1}(l_{n,i}) \right)^k \prod_{i=1}^{n} q_{r-1}(l_{n,i}) = (-1)^{nk(n+1)/2} \prod_{i=1}^{n} q_{r-1}(l_{n,i})\end{aligned}$$

by above. Note that $\gcd(n+1, m+1) = \gcd(r, n+1)$, by construction. Further, the remainder when $(n+1)$ is divided by r is less than r. Thus, by our induction hypothesis, if $\gcd(n+1, m+1) \neq 1$, then $\gcd(r, n+1) \neq 1$, so,

$$\prod_{i=1}^{n} q_m(l_{n,i}) = (-1)^{nk(n+1)/2} \prod_{i=1}^{n} q_{r-1}(l_{n,i}) = 0.$$

Otherwise, $\gcd(n+1, m+1) = 1$, so $\gcd(r, n+1) = 1$, implying that

$$\begin{aligned}\prod_{i=1}^{n} q_m(l_{n,i}) &= (-1)^{nk(n+1)/2} \prod_{i=1}^{n} q_{r-1}(l_{n,i}) = (-1)^{nk(n+1)/2} (-1)^{n(r-1)/2} \\ &= (-1)^{n(k(n+1)+r-1)/2} = (-1)^{nm/2}.\end{aligned}$$

□

The following corollary to theorem 2.6 follows immediately.

Corollary 2.7. *Suppose n and m are positive integers. Then,*

$$\det\left(\mathbf{A}(P_n \underline{P}_m)\right) = \begin{cases} 0 & \textit{if } \gcd(n+1, m+1) \neq 1, \\ (-1)^{nm/2} & \textit{if } \gcd(n+1, m+1) = 1. \end{cases}$$

□

References

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